Ground State of the Quantum Symmetric Finite Size XXZ Spin Chain with Anisotropy Parameter $\Delta = \frac{1}{2}$

V. Fridkin[a], Yu. Stroganov[a,b] and D. Zagier[c]

[a] Research Institute for Mathematical Sciences
Kyoto University, Kyoto 606, Japan
[b] Institute for High Energy Physics
Protvino, Moscow region, Russia
[c] Max-Planck-Institut fur Mathematik
Gottfried-Claren-Strasse 26, D-53225, Bonn, Germany

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Abstract

We find an analytic solution of the Bethe Ansatz equations (BAE) for the special case of a finite XXZ spin chain with free boundary conditions and with a complex surface field which provides for $U_q(sl(2))$ symmetry of the Hamiltonian. More precisely, we find one nontrivial solution, corresponding to the ground state of the system with anisotropy parameter $\Delta = \frac{1}{2}$ corresponding to $q^3 = -1$.

Dedicated to Rodney Baxter
on the occasion of his 60th birhday.

It is widely accepted that the Bethe Ansatz equations for an integrable quantum spin chain can be solved analytically only in the thermodynamic limit or for a small number of spin waves or short chains. In this letter, however, we have managed to find a special solution of the BAE for a spin chain of arbitrary length N with N/2 spin waves.

It is well known (see, for example [1] and references therein) that there is a correspondence between the Q-state Potts Models and the Ice-Type Models with anisotropy parameter $\Delta = \frac{\sqrt{Q}}{2}$. The coincidence in the spectrum of an N-site self-dual Q-state quantum Potts chain with free ends with a part of the spectrum of the $U_q(sl(2))$ symmetrical 2N-site XXZ Hamiltonian (1) is to some extent a manifestation of this correspondence.

$$H_{xxz} = \sum_{n=1}^{N-1} \{\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+ + \frac{q+q^{-1}}{4}\, \sigma_n^z \sigma_{n+1}^z \; + \frac{q-q^{-1}}{4}\, (\sigma_n^z - \sigma_{n+1}^z)\}, \tag{1}$$

where $\Delta = (q + q^{-1})/2$. This Hamiltonian was considered by Alcaraz *et al.* [1] and its $U_q(sl(2))$ symmetry was described by Pasqier and Saleur [2]. The family of commuting transfer-matrices that commute with H_{xxz} was constructed by Sklyanin [3] incorporating a method of Cherednik [4].

Baxter's T-Q equation for the case under consideration can be written as [5]

$$t(u)Q(u) = \phi(u + \eta/2)Q(u - \eta) + \phi(u - \eta/2)Q(u + \eta) \tag{2}$$

where $q = \exp i\eta$, $\phi(u) = \sin 2u \sin^{2N} u$ and $t(u) = \sin 2u\, T(u)$. The $Q(u)$ are eigenvalues of Baxter's auxilary matrix $\hat{Q}(u)$, where $\hat{Q}(u)$ commutes with the transfer matrix $\hat{T}(u)$. The eigenvalue $Q(u)$ corresponding to an eigenvector with $M = N/2 - S_z$ reversed spins has the form

$$Q(u) = \prod_{m=1}^{M} \sin(u - u_m) \sin(u + u_m). \tag{3}$$

Equation (2) is equivalent to the Bethe Ansatz equations [6]

$$\left[\frac{\sin(u_k + \eta/2)}{\sin(u_k - \eta/2)}\right]^{2N} = \prod_{m \neq k}^{M} \frac{\sin(u_k - u_m + \eta)\sin(u_k + u_m + \eta)}{\sin(u_k - u_m - \eta)\sin(u_k + u_m - \eta)}. \tag{4}$$

In a recent article [9] it was argued that the criteria for the above mentioned correspondence is the existence of a second trigonometric solution for Baxter's T-Q equation and it was shown that in the case $\eta = \pi/4$ the spectrum of H_{xxz} contains the spectrum of the Ising model. In this article we limit ourselves to the case $\eta = \pi/3$. This case is in some sense trivial since for this value of η, H_{xxz} corresponds to the 1-state Potts Model. We find only one eigenvalue $T_0(u)$ of the transfer-matrices $\hat{T}(u)$ when Baxter's equation (2) has two independent trigonometric solutions. Solving for $T(u) = T_0(u)$ analytically we find a trigonometric polynomial $Q_0(u)$ the zeros of which satisfy the Bethe Ansatz equations (4). The number of spin waves is equal to $M = N/2$. The corresponding eigenstate is the groundstate of H_{xxz} with eigenvalue $E_0 = \frac{3}{2}(1 - N)$, as discovered by Alcaraz *et al.* [1] numerically.

When does a second independent periodic solution exist? This question was considered in article [9]. Here we use a variation more convenient for our goal.

Let us consider T-Q equation (2) for $\eta = \frac{\pi}{L}$, where $L \geq 3$ is an integer. Let us fix a sequence of spectral parameter values $v_k = v_0 + \eta k$, where k are integers and write $\phi_k = \phi(v_k - \eta/2)$, $Q_k = Q(v_k)$ and $t_k = t(v_k)$. The functions $\phi(u)$, $Q(u)$ and $t(u)$ are periodic with period π. So the sequences we have introduced are also periodic with period L, i.e., $\phi_{k+L} = \phi_k$, etc..

Setting $u = v_k$ in (2) gives the linear system

$$t_k Q_k = \phi_{k+1} Q_{k-1} + \phi_k Q_{k+1}. \tag{5}$$

The matrix of coefficients for this system has a tridiagonal form. Taking $v_0 \neq \frac{\pi m}{2}$, where m is an integer, we have $\phi_k \neq 0$ for all k.

It is straightforward to calculate the determinant of the $L - 2 \times L - 2$ minor obtained by deleting the two left most columns and two lower most rows. It is equal to the product $-\phi_1^2 \phi_2 \phi_3 \dots \phi_{L-1}$, which is nonzero, hence the rank of M cannot be less than $L - 2$. Here we are interested in the case when the rank of M is precisely $L-2$ and we have two linearly independent solutions for equation (5). Let us consider the three simplest cases L = 3, 4 and 5. The parameter η is equal to $\frac{\pi}{3}$, $\frac{\pi}{4}$ and $\frac{\pi}{5}$ respectively.

For $L=3$ the rank of M is unity and we immediately get $t_0=-\phi_2$, $t_1=-\phi_0$ and $t_2=-\phi_1$. Returning to the functional form, we can write

$$T_0(u)=t_0(u)/\sin 2u=-\phi(u+\pi/2)/\sin 2u=\cos^{2N}u. \tag{6}$$

This is the unique eigenvalue of the transfer-matrix for which the T-Q equation has two independent periodic solutions. It is well known (see, for example, [6]) that the eigenvalues of H_{xxz} are related to the eigenvalues $t(u)$ by

$$E=-\cos\eta(N+2-\tan^2\eta)+\sin\eta\frac{t'(\eta/2)}{t(\eta/2)}. \tag{7}$$

For the eigenstate corresponding to eigenvalue (6) we obtain $E_0=3/2(1-N)$. This is the groundstate energy which was discovered by Alcaraz *et al.* [1] numerically.

Below we find all solutions of Baxter‘s T-Q equation corresponding to $T(u)=T_0(u)$. Zeros of these solutions satisfy the BAE (4). In particular we find $Q_0(x)$ corresponding to physical Bethe state.

For $L=4$, deleting the second row and the forth column of M we obtain a minor with determinant $-\phi_0\phi_3(t_0+t_2)$. It is zero when $t_2=-t_0$, i.e., $t(u+\frac{\pi}{2})=-t(u)$. Considering the other minors we obtain the functional equation

$$t(u+\pi/8)t(u-\pi/8)=\phi(u+\pi/4)\phi(u-\pi/4)-\phi(u)\phi(u+\pi/2). \tag{8}$$

This functional equation was used in [9] to find $t(u)$ and show that this part of the spectrum of H_{xxz} coincides with the Ising model. It would be interesting to find a corresponding $Q(u)$.

Lastly for $L=5$, minor M_{35} (the third row and the fifth column are deleted) has determinant $\phi_0\phi_4(t_0t_1+\phi_1t_3-\phi_0\phi_2)$. Setting this to zero we have

$$t(u)t(u+\pi/5)+\phi(u+\pi/10)t(u+3\pi/5)-\phi(u-\pi/10)\phi(u+3\pi/10)=0. \tag{9}$$

It is not difficult to check that in this case all 4×4 minors have zero determinant and that the rank of M is 3. Thus we have two independent periodic solutions of Baxter's T-Q equation.

Note that this functional relation coincides with the Baxter-Pearce relation for the hard hexagon model [10]. We have obtained the same truncated functional relations that have been obtained in [9] with the same assumptions.

We now consider the solution of Baxter's Equation for $\eta=\frac{\pi}{3}$ and $T=T_0$. For $\eta=\frac{\pi}{3}$ and transfer-matrix eigenvalue $T_0(u)=\cos^{2N}u$, the T-Q equation (2) reduces to

$$\phi(u+3\eta/2)Q(u)+\phi(u-\eta/2)Q(u+\eta)+\phi(u+\eta/2)Q(u-\eta)=0. \tag{10}$$

This equation can be rewritten as

$$f(v)+f(v+2\pi/3)+f(v+4\pi/3)=0, \tag{11}$$

where $f(v) = \sin v \, \cos^{2N}(v/2)\, Q(v/2)$ has period 2π. The trigonometric polynomial $f(v)$ is an odd function, so it can be written

$$f(v) = \sum_{k=1}^{K} c_k \sin kv, \tag{12}$$

where K is the degree of $f(v)$. Then equation (11) is equivalent to $c_{3m} = 0$, $m \in Z$.

The condition that $f(v)$ be divisible by $\sin v \cos^{2N}(v/2)$ is equivalent to

$$(\frac{d}{dv})^i f(v)|_{v=\pi} = 0, \qquad i = 0, 1, \ldots, 2N. \tag{13}$$

For even i this condition is immediate, whereas for $i = 2j - 1$ we use (12) to obtain

$$\sum_{k=1, k\neq 3m}^{K} (-1)^k c_k k^{2j-1} = 0, \qquad j = 1, 2, \ldots, N. \tag{14}$$

Our problem is thus to find $\{c_k\}$ satisfying the last equation. This problem is a special case of a more general problem which can be formulated as follows. Given a set of different complex numbers $X = \{x_1, x_2, \ldots, x_I\}$ we seek another complex set $B = \{\beta_1, \beta_2, \ldots, \beta_I\}$ where $\beta_i \neq 0$ for some i, so that

$$\sum_{i=1}^{I} \beta_i P(x_i) = 0 \tag{15}$$

for any polynomial $P(x)$ of degree not more than $N - 1$. It is clear that for $I \leq N$ the system B does not exist. If $\beta_1 \neq 0$, for example, the product $(x - x_2)(x - x_3)\ldots(x - x_I)$ provides a counterexample.

Let $I = N + 1$. We try the polynomials

$$P_r = \prod_{i=1, i\neq r,}^{N} (x - x_i), \qquad r = 1, 2, \ldots, N. \tag{16}$$

Condition (15) gives $\beta_r P_r(x_r) + \beta_I P_r(x_I) = 0$ and we immediately obtain

$$\beta_r = \mathrm{const} \prod_{i=1, i\neq r}^{N+1} (x_r - x_i)^{-1}, \tag{17}$$

which is a solution because the system (16) forms a basis of the linear space of $N-1$ degree polynomials. So for $I = N + 1$ we have a unique solution (up to an arbitrary nonzero constant) given by (17). It is easy to show that for $I = N + \nu$ we obtain a ν-dimensional linear space of solutions.

Returning to (14) we consider $N = 2n$, n a positive integer. Fix $I = N + 1 = 2n + 1$. The degree K becomes $3n + 1$. It is convenient to use a new index $k = |3\kappa + 1|$, where $|\kappa| \leq n$. Equation (14) can be rewritten as

$$\sum_{\kappa=-n}^{n} \beta_\kappa (3\kappa + 1)^{2(j-1)} = 0, \qquad j = 1, 2, \ldots, N, \tag{18}$$

where we use new unknowns $\beta_\kappa = (-1)^\kappa c_{|3\kappa+1|}|3\kappa+1|$ instead of c_k. Using (17) and (12) we obtain the function $f(v)$

$$f(v) = \sum_{\kappa=-n}^{n} (-1)^\kappa \binom{2n+\frac{2}{3}}{n-\kappa} \binom{2n-\frac{2}{3}}{n+\kappa} \sin(3\kappa+1)v. \tag{19}$$

We recall that the solution of Baxter's T-Q equation for $T(u) = T_0(u)$ is given by

$$Q_0(u) = f(2u)/(\sin 2u \cos^{2N} u) \tag{20}$$

and its zeros $\{u_k\}$ satisfy the BAE (4).

Another way to derive the above solution is to observe that the function $f(v)$ satisfies a simple second order linear differential equation. Indeed, it is easily seen that the functions $F^+(x)$ and $F^-(x)$, where

$$F^+(x) = \sum_{\kappa=-n}^{n} (-1)^\kappa \binom{2n+\frac{2}{3}}{n-\kappa} \binom{2n-\frac{2}{3}}{n+\kappa} x^{\kappa+\frac{1}{3}} \text{ and } F^-(x) = F^+(1/x). \tag{21}$$

are the two linearly independent solutions of the differential equation

$$\{((\theta+n)^2 - 1/9)/x + (\theta-n)^2 - 1/9\}F^+ = 0, \tag{22}$$

where $\theta = x\frac{d}{dx}$.[1] Now the fact that there is a combination $f(v)$ of $F^+(e^{3iv})$ and $F^-(e^{3iv})$ which vanishes to order $2N+1$ at $v = \pi$ follows immediately from the fact that $x = -1$ is a singular point of the differential equation (22) and that the indicial equation at this point has roots 0 and $2n+1$. In terms of the variable v, equation (22) becomes

$$\frac{d^2 f}{dv^2} + 6n \tan(3v/2)\frac{df}{dv} + (1-9n^2)\, f = 0. \tag{23}$$

The zeros of $f(v)$, the density of which is important in the thermodynamic limit, are located on the imaginary axis in the complex v-plane. So it is convenient to make the change of variable $v = is$. It is also useful to introduce another function $g(s) = f(is)/\cosh^{2n}(3s/2)$. The differential equation for $g(s)$ is then

$$g'' + \left(\frac{9n(2n+1)}{2\cosh^2(3s/2)} - 1\right) g = 0. \tag{24}$$

Let $g(s_0) = 0$. For large n we have in a small vicinity of s_0 an approximate equation $g'' + \omega_0^2 g = 0$. This equation describes a harmonic oscillator with frequency $\omega_0 = 3n/\cosh(3s_0/2)$. The distance between nearest zeros is approximately $\Delta s = \pi/\omega$ and we obtain the following density function which describes the number of zeros per unit length

$$\rho(s) = 1/\Delta s = \omega/\pi = 3n/(\pi \cosh(3s/2)). \tag{25}$$

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[1] Up to a change of variables this is just the standard hypergeometric differential equation, and in fact $F^+(x) = \text{const } F(-2n, 2/3-2n, 5/3, -x)x^{1/3-n}$

We note that equation (24) has a history as rich as the BAE. Eckart [11] used the Schrodinger equation with bell-shaped potential $V(r) = -G/\cosh^2 r$ for phenomenological studies in atomic and molecular physics. Later it was used in chemistry, biophysics and astrophysics, just to name a few. For more recent references see, for example, [12].

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